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Dynamics of the Infinite-Ranged Potts Model

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We formulate a theory of single-spin-flip dynamics for the infinite-range q-state Potts model. We derive a Fokker–Planck equation, without diffusive term, from a phenomenological master equation. It describes the approach to equilibrium of the time-dependent probability density and thus generalizes Griffiths' (1966) result for the Ising model. We particularly compare the dynamic evolutions of $q=2$ and $q=3$ systems when sinusoidal external fields are applied. In the case $q=2$ we find evidence of a nonequilibrium phase transition and for $q=3$ period doubling bifurcations are observed, yielding a good estimate of Feigenbaum's universal exponent.

KEY WORDS: Glauber dynamics; Potts model; Fokker–Planck equation; relaxation; period-doubling.

1. INTRODUCTION

The Potts model[4,8,12] (see ref. 13 for a review) is perhaps the simplest generalization of the Ising model. It is the purpose of this paper to investigate the dynamic evolution of the general q-state Potts model in the infinite-range limit. This work is essentially a generalization of Griffith's[16] results for the Ising model.

We deduce a Fokker–Planck[11] equation for the probability distribution of the order parameter components, from a master equation which describes single-spin-flip processes with rates chosen to obey detailed balance and reducible to Glauber's[5] in the limit $q=2$. We also add to the Hamiltonian a periodic external magnetic field which drives the system away from equilibrium.

This paper is organized as follows. In the next section we define the model and present our basic equations. In Section 3 we study the dynamic

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evolution. Time-dependent effects are presented in Section 4 for the $q=2$ and $q=3$ cases. In the Appendix we show in more detail the intermediary calculations leading to the final result.

2. THEORY. THE DYNAMIC MODEL

We consider the q-state Potts model described by the following Hamiltonian with N spin variables σ_i ($\sigma_i \in \{1, 2,..., q\}$):

$$\beta \mathscr{H}_N(\sigma_N) = -q \frac{\beta J}{N} \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j} - q \sum_{\alpha} \sum_{i} h_\alpha \delta_{\sigma_i, \alpha} \tag{1}$$

Here the second term on the right side is the Zeeman energy, and the exchange couples every pair of spins. Since this model has no intrinsic dynamics, we define a stochastic evolution[5] with single spin flips, induced through contact with a thermal bath. With $P(\{\sigma\}; t)$ denoting the probability for the configuration $\{\sigma\} = \{\sigma_1, \sigma_2,..., \sigma_N\}$ to be realized at time t, and $W(\sigma_i, \{\sigma\}_i | \sigma_i', \{\sigma\}_i)$ the rate for the system to change the state of the spin i from σ_i to σ_i', we easily write the phenomenological master equation as

$$\begin{aligned}\frac{\partial P(\{\sigma\}; t)}{\partial t} = &\sum_i \sum_{\{\sigma_i'\}} W(\sigma_i', \{\sigma\}_i | \sigma_i, \{\sigma\}_i)\, P(\sigma_i', \{\sigma\}_i; t) \\ &- \sum_i \sum_{\{\sigma_i'\}} W(\sigma_i, \{\sigma\}_i | \sigma_i', \{\sigma\}_i)\, P(\sigma_i, \{\sigma\}_i; t)\end{aligned} \tag{2}$$

and $P(\{\sigma\}; t)$ obeys $\sum_{\{\sigma\}} P(\{\sigma\}; t) = 1$.

This master equation describes the system with N spins and with q^N states. In the following we use a single-spin-flip rate[8] which was shown to obey detailed balance and to correctly reduce to Glauber's form in the appropriate limit:

$$W(\sigma_i \to \sigma_i') = \frac{\exp(\beta \mathscr{H}_N(\sigma_i))}{\sum_{\{\sigma_i = 1, 2,..., q\}} \exp(\beta \mathscr{H}_N(\sigma_i))} \tag{3}$$

Others choices of rates can be used,[4,12] but our final results [Eq. (6)] are affected only by a multiplicative factor, which does not introduce any qualitative differences in the evolution to equilibrium.

We define x_σ to be the fraction of spins in the σ state:

$$x_\sigma = \frac{n_\sigma}{N} = \frac{1}{N} \sum_{j=1}^{N} \delta_{\sigma, \sigma_j}, \qquad \sigma = 1, 2,..., q \tag{4}$$

Obviously, we obtain the sum rule $x_1 + \cdots + x_q = 1$, which implies that the state of the system is confined to this $(q-1)$-dimensional space. These fractions play the role of order parameter components.[13] We now consider the probability density $P(x_1, x_2, ..., x_q; t) \equiv P(\mathbf{x}; t)$:

$$P(x_1, x_2, ..., x_q; t) = \sum_{\{\sigma\}} P(\sigma_1, \sigma_2, ..., \sigma_N; t) \prod_{\alpha=1}^{q} \delta_{x_\alpha,\, (1/N)\sum_{j=1}^{N} \delta_{\alpha, \sigma_j}} \tag{5}$$

After some tedious but straightforward algebra (see the Appendix) we obtain a Fokker–Planck equation without diffusive term. This thus yields a deterministic evolution of the system which is equivalent to Langevin's flow equations for the order parameter components:

$$\frac{dx_\alpha(t)}{dt} = \sum_{\sigma=1}^{q} (1 - q\delta_{\alpha,\sigma}) W_\sigma x_\sigma, \qquad \alpha = 1, ..., q \tag{6}$$

where

$$W_\sigma = e^{-kx_\sigma} \Big/ \sum_{\alpha=1}^{q} e^{-kx_\alpha}, \qquad k = \beta J \tag{7}$$

These $q-1$ coupled nonlinear equations are our final result. They will be shown to exhibit the decay to equilibrium of the $q-1$ independent modes. If we assume that a small external field is applied along a particular direction in parameter space (say, direction $\alpha = 1$), we may introduce the order parameter (longitudinal mode) $m(t)$ and the transverse modes $\varepsilon_i(t)$, through

$$x_i(t) = \frac{1}{q}[1 + m(t)(q\delta_{i,1} - 1)] + \varepsilon_i(t) \tag{8}$$

Then we easily find that the order parameter evolution decouples from the others and it shows how special the Ising model is (since it does not present the transverse modes):

$$\frac{dm}{dt} = -\left[\frac{1 + (q-1)e^{-km}}{e^{-km} + q - 1} m + \frac{e^{-km} - 1}{e^{-km} + q - 1}\right] \tag{9}$$

We always find the transverse modes to present fast relaxation, while the slow, longitudinal mode is the only one to exhibit critical slowing down. Equation (9) can be written in the following form:

$$\frac{\partial m}{\partial t} = -T(k, m) \frac{\partial G(k, m)}{\partial m} \tag{10}$$

where $T(k, m)$ is given by

$$T(k, m) = \frac{1 + (q-1)e^{-kqm}}{e^{-kqm} + q - 1}$$

and $G(k, m)$ by

$$G(k, m) = \frac{m^2}{2} - \frac{1}{qk} \ln(e^{kqm} + q - 1) - \frac{1}{qk} \ln[e^{kqm}(q-1) + 1] \qquad (11)$$

We easily can see that $T(k, m)$ is always positive and $G(k, m)$ is a non-increasing function of time:

$$\frac{\partial G(k, m)}{\partial t} \leqslant 0$$

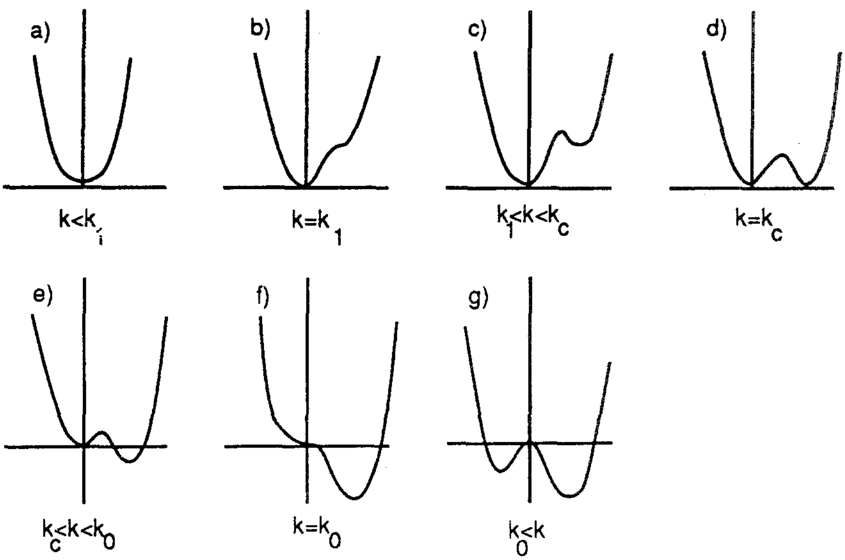


Fig. 1. Schematic representation of the free energy as a function of the order parameter m, for different values k (inverse temperature). (a) Paramagnetic region I. The only minimum at $m = 0$ indicates one stable paramagnetic phase. (b) Upper spinodal temperature. The inflection point indicates a double solution to an unstable ordered phase. (c) Paramagnetic region II. The local minima or maxima represent the metastable and unstable phases, respectively. (d) First-order phase transition (para-ferromagnetic). (e). Ordered region I. The paramagnetic phase is metastable. (f) Lower spinodal temperature. The inflection point is located at $m = 0$. The paramagnetic susceptibility presents a divergence here. (g) Ordered region II. Two minima indicate a stable phase ($m > 0$) and a metastable ($m < 0$) phase. The paramagnetic phase is unstable.

Here, the equality holds if and only if an equilibrium state has been reached. Thus $G(k, m)$ is essentially a Lyapunov function which contains the entire stability structure of the dynamic system and the system will always approach one of the solutions of

$$m - \frac{e^{kqm} - 1}{e^{kqm} + q - 1} = 0$$

Analysis of the local stability of these solutions can be carried out on the usual way by linearizing Eq. (9) around the equilibrium point. This leads, as expected, to the well-known mean-field critical dynamic behavior and it shall be omitted here for conciseness. We are more interested in the general behavior of the system where the influence of the nonlinear terms are important. In order to further study the solutions of Eq. (9), we have made some numerical investigations.

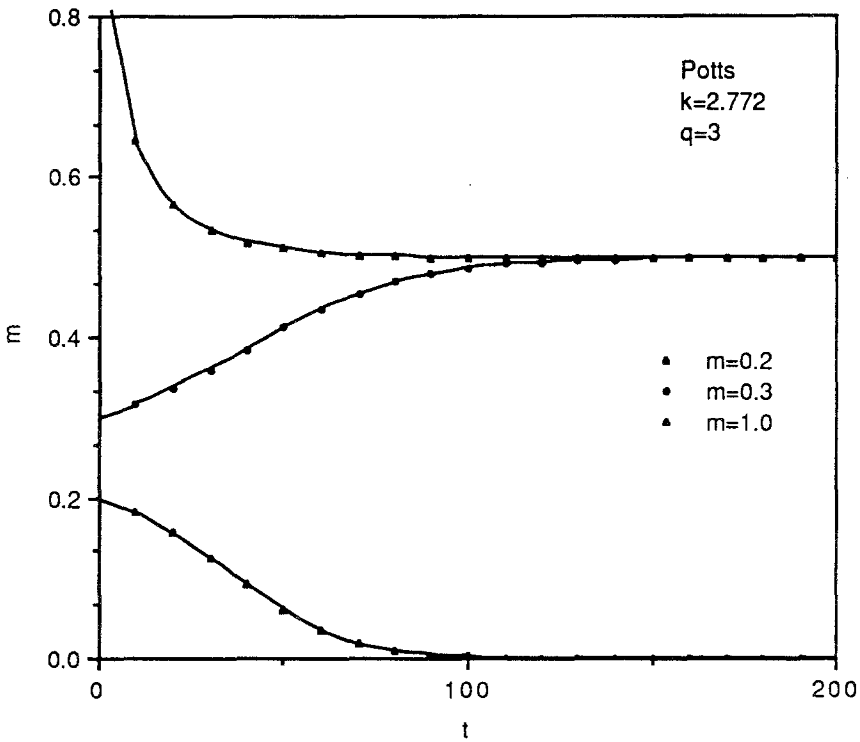


Fig. 2. Evolution to equilibrium for $q=3$ and $k=2.772$, with differents starting point m_0 (0.2, 0.3, 1.0).

3. DYNAMIC EVOLUTION

In this section we present and discuss the solution of Eqs. (6) and (9) for the case $q=3$. In the mean field approximation (see ref. 13 for details) the Potts model has a first-order phase transition, with a discontinuity in the order parameter $\Delta m = 2\ln(q-1)$ at k_C. If we analyze the free energies of this model (Fig. 1 shows the free energy along a symmetry-broken direction for different temperatures), we can define three relevant (inverse) temperatures: k_1, upper spinodal temperature (stability limit of the ordered phase), k_C, the temperature of the first-order phase transition (equality of the free energies), and k_0, the lower spinodal temperature (stability limit of the paramagnetic phase). For $q=3$, they are given by $k_1=2.7456$, $k_C=2.7725$, $k_0=3.0$.[13]

We study numerically the evolution given by Eq. (9) in different domains of temperature. Figure 2 shows the evolution to the minima of the free energy for different initial values m_0 and in Fig. 3 we see that even if

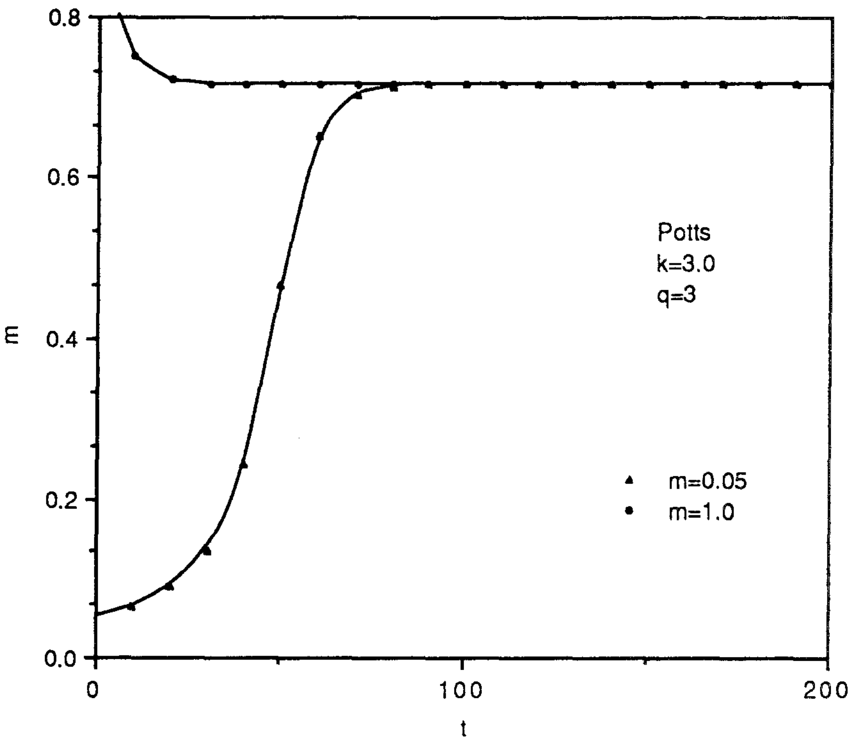


Fig. 3. Evolution to equilibrium for $q=3$ and $k=3.0$, with differents starting points m_0 (0.05, 1.0).

the system starts near one of the metastable points, which lie closer to the origin, it will evolve toward a stable point (as can also be seen from Fig. 6). Therefore, the metastable points are actually saddle points in configuration space. Without symmetry breaking the evolution of the $q=3$ system described by Eq. (6) is better expressed in the plane defined by $x_1+x_2+x_3=1$, where the free energy has several minima and maxima as a function of k. These points can be seen in Figs. 5–6. This space is more conveniently described by the symmetrical combinations:

$$Q_1(t)=\frac{\sqrt{3}}{2}(x_2-x_1) \qquad \text{and} \qquad Q_2(t)=1-\frac{3}{2}(x_1+x_2) \tag{12}$$

The solutions of Eq. (6) are then expressed as flux diagrams. We choose initial values very close to the boundaries; Figs. 4–6 show the evolution for

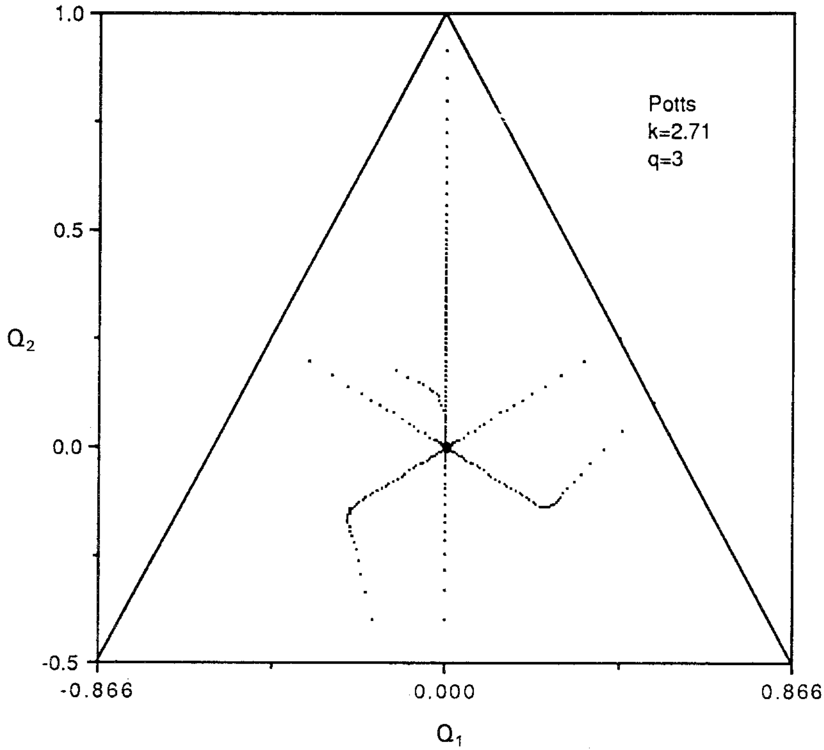


Fig. 4. Trajectories in the Q_1–Q_2 plane (for $q=3$ Potts model) for different initial starting points. Tick marks indicate equal time intervals.

different values of k. The phase space has the symmetry of an equilateral triangle; therefore the different starting points plotted in the graph can be rotated to give more detailed information.

At high temperatures ($k < k_1$, Fig. 4, Fig. 1a) the system evolves to the origin, independently of the starting point. For $k = k_C$ (Fig. 5, Fig. 1d) the situation is qualitatively different, since now the system has four attractors, with basins of attraction clearly isolated by separatrices. In the low-temperature phase ($k > k_0$, Fig. 6, Fig. 1g) there are six fixed points, but three of them correspond to saddle points which can only be reached if the starting point lies at one of the bisections of the triangle. In all other cases the system goes to one of one of the absolute minima (stable points), which lie nearer to the vertices of the triangle. These results, which, of course, conform with the expected equilibrium phases of the system, were obtained by computer integration, using the fourth-order Runge-Kutta method with adaptive stepwise.

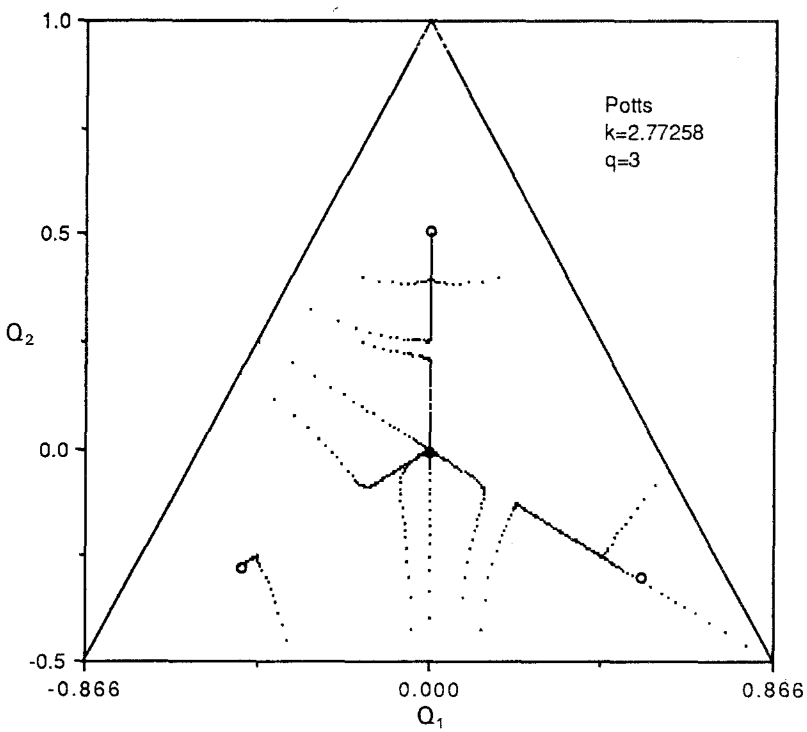


Fig. 5. Trajectories in the Q_1–Q_2 plane (for $q = 3$ Potts model) for different initial starting points. Tick marks indicate equal time intervals. Open circles correspond to stable states and full circles to metastable solutions, for $k = 2.77258$.

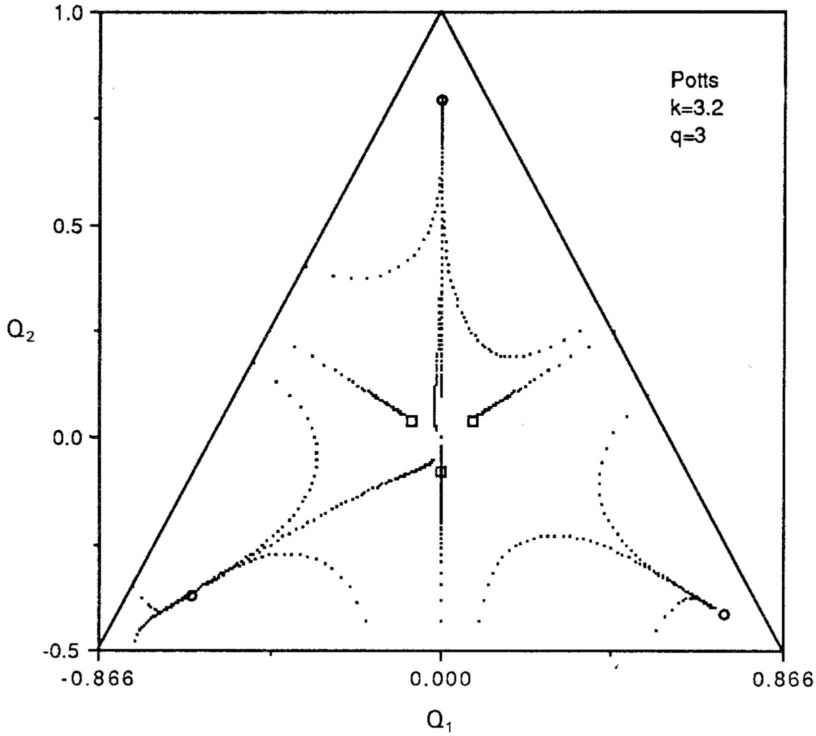


Fig. 6. Trajectories in the Q_1–Q_2 plane (for $q = 3$ Potts model) for different initial starting points. Tick marks indicate equal time intervals. Open circles correspond to stable states and the squares to metastable solutions, for $k = 3.2$.

4. TIME-DEPENDENT EFFECTS

4.1. Ising Model ($q = 2$)

We consider the dynamic evolution under the influence of a time-dependent external magnetic field:

$$h(t) = h_0 \sin(\omega t) \tag{13}$$

When the ac magnetic field is absent our results are the same as derived by others authors.[1,6,10] With the field, our equation of motion takes the form

$$\frac{dm}{dt} = -m(t) + \tanh[E(t)] \tag{14a}$$

$$E(t) = km(t) + h(t) \tag{14b}$$

We want to study how the phase diagram changes with the amplitude and frequency of the applied magnetic field. It is here useful to introduce the static spinodal ("coercive field") magnetic field h_C, given by

$$h_C = \frac{1}{2}\ln(1+m_C) - \frac{1}{2}\ln(1-m_C) - Tm_C, \quad \text{where } m_C = \left(1 - \frac{1}{T}\right)^{1/2} \quad (T>1) \tag{15}$$

This is the field above which there is only one minimum for the free energy and it corresponds to the transformation of a metastable state into an unstable one. For small $m(t)$, $h(t)$, and $E(t)$ we can analyze Eq. (14a) by expanding its right-hand side in powers of $E(t)$. However, we are interested also in the case of external ac field with strong amplitude, and this compels us to solve Eq. (14a) numerically; the solution $m(t)$ after a stationary state has been reached is decomposed in Fourier series:

$$m(t) = \frac{m_0}{2} + \sum_{n=1}^{\infty} m_n \sin(n\omega t + \theta_n) \tag{16}$$

One notices that the amplitudes of the first harmonics (m_1 and m_2) are proportional to the linear and nonlinear susceptibilities. Figures 7a–7c

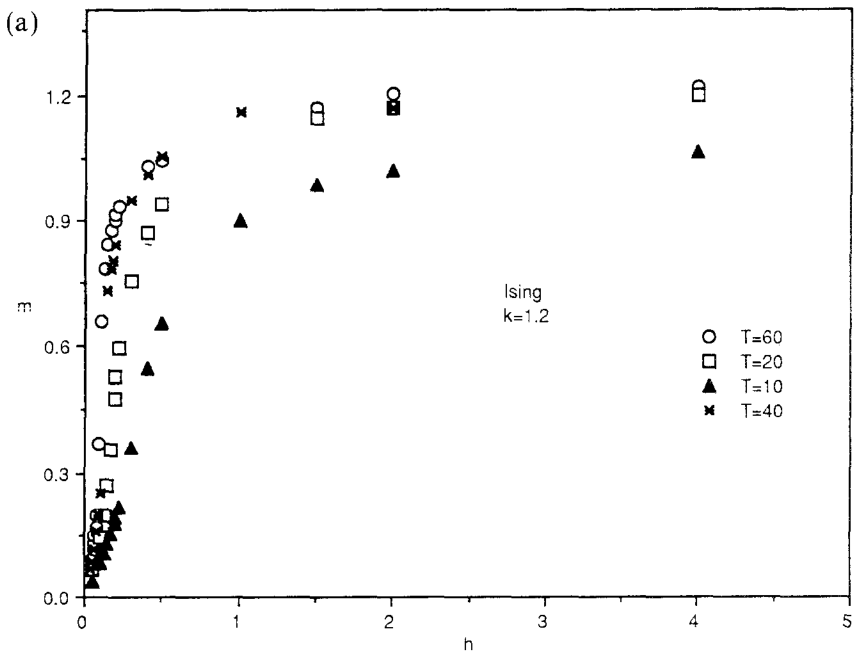
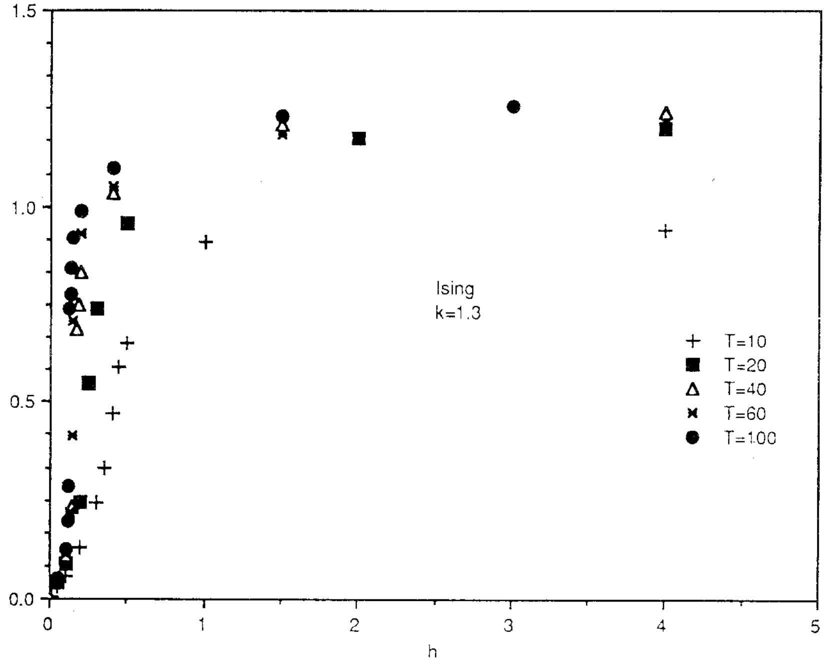


Fig. 7. Representation of the first harmonic for differents values of the applied period T ($q=2$). The sharp increase appears for distinct values of h_0. (a) $k=1.2$; (b) $k=1.3$; (c) $k=1.5$.

(b)



(c)

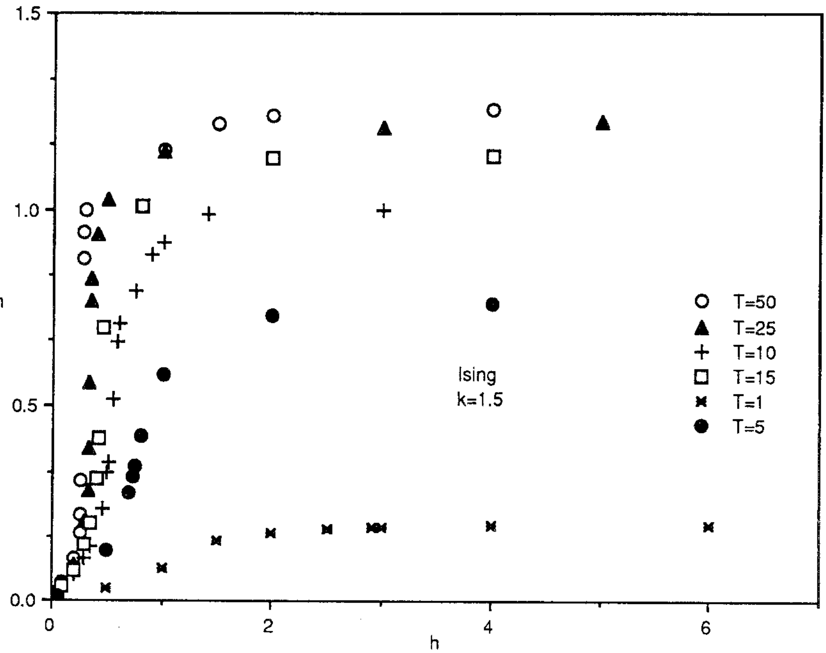


Fig. 7. (*Continued*)

show the amplitudes of the first harmonic for different values of the external period and amplitude h_0. We find that there appears to exist a well-defined field amplitude h'_c for which the first harmonic has a sharp increase and the second harmonic (m_2) presents a maximum. We show in Figs. 8a–8c the amplitudes m_0, m_1, m_2. We thus identify an out-of-equilibrium dynamic transition (which generalizes the static, out-of-equilibrium transition taking place at $h = h_c$, for the case of a static magnetic field). In Fig. 9 we plot the frequency dependence of this critical field h'_c; we find that the law

$$h'_c - h_C \propto \frac{1}{T} \propto \omega$$

is well obeyed not only at small frequencies (where the adiabatic approximation should hold), but at higher frequencies as well.

We interpret these results as follows: in the static case, for fields below

(a)

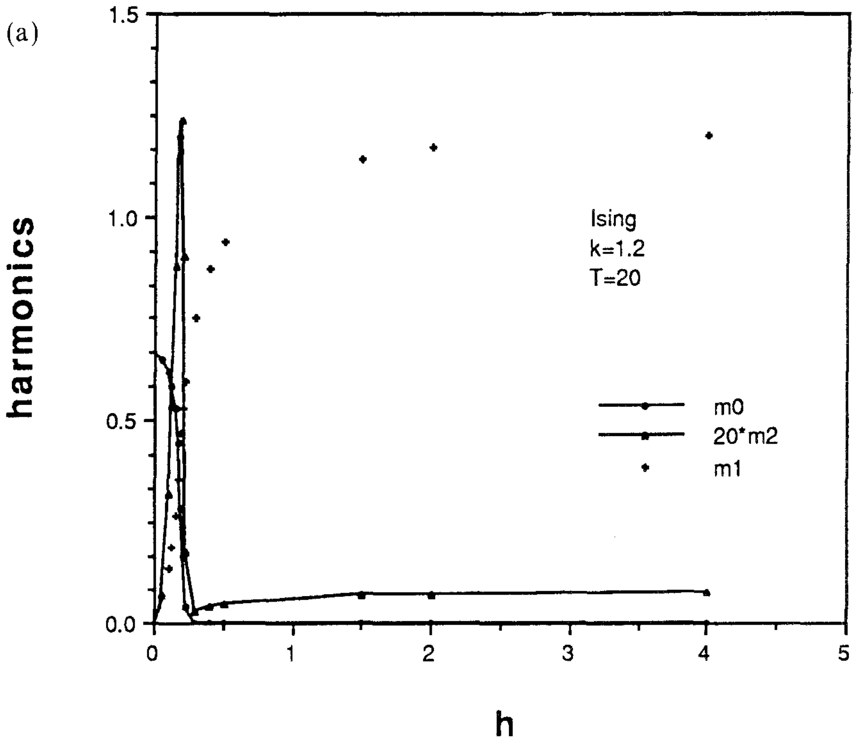
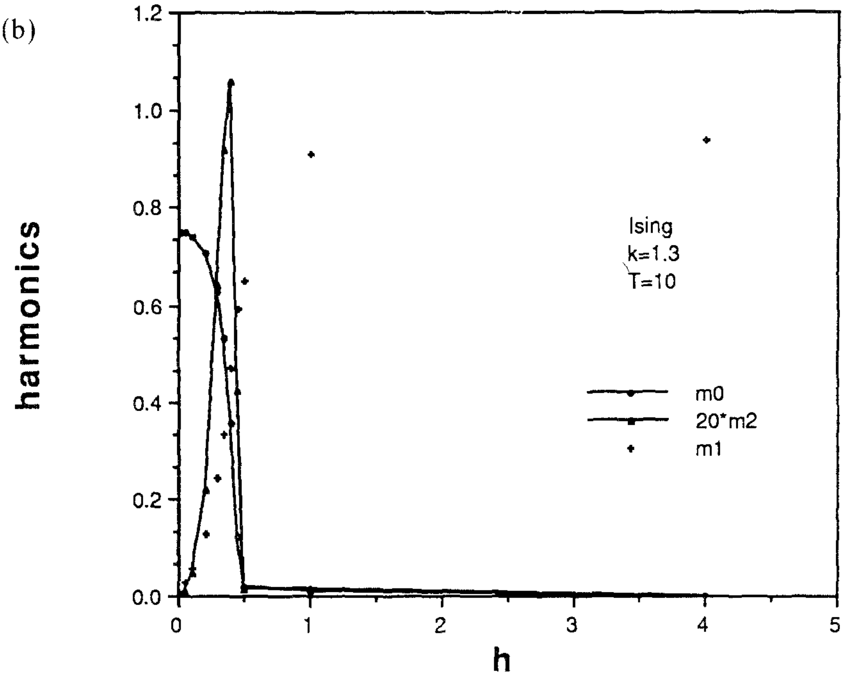


Fig. 8. Representation of the three first harmonics m_0, m_1, and m_2 for different values of temperature k and period T. (a) $k = 1.2$, $T = 20$; (b) $k = 1.3$, $T = 10$; (c) $k = 1.5$, $T = 50$.

(b)



(c)

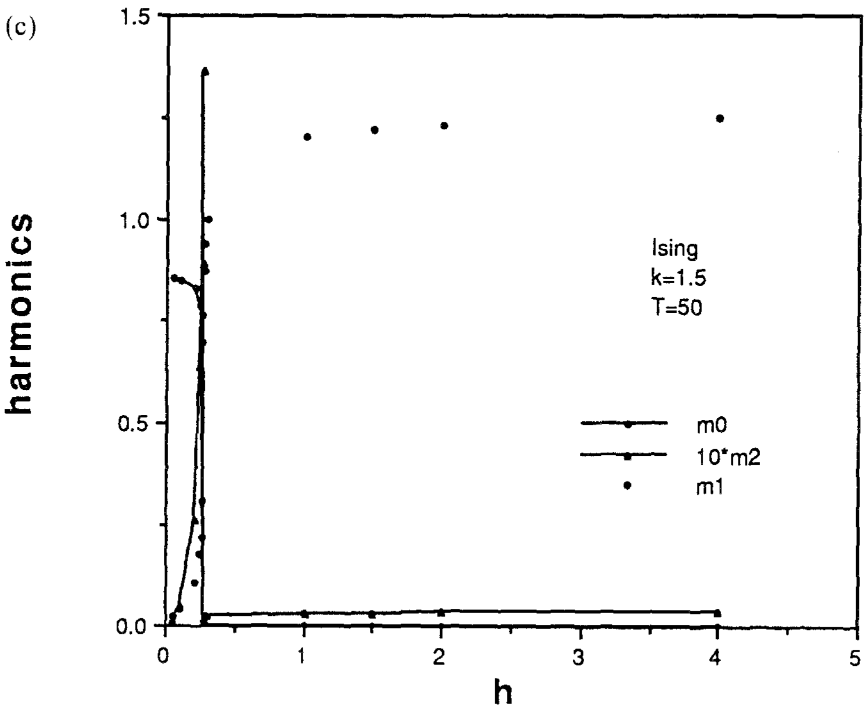


Fig. 8. (*Continued*)

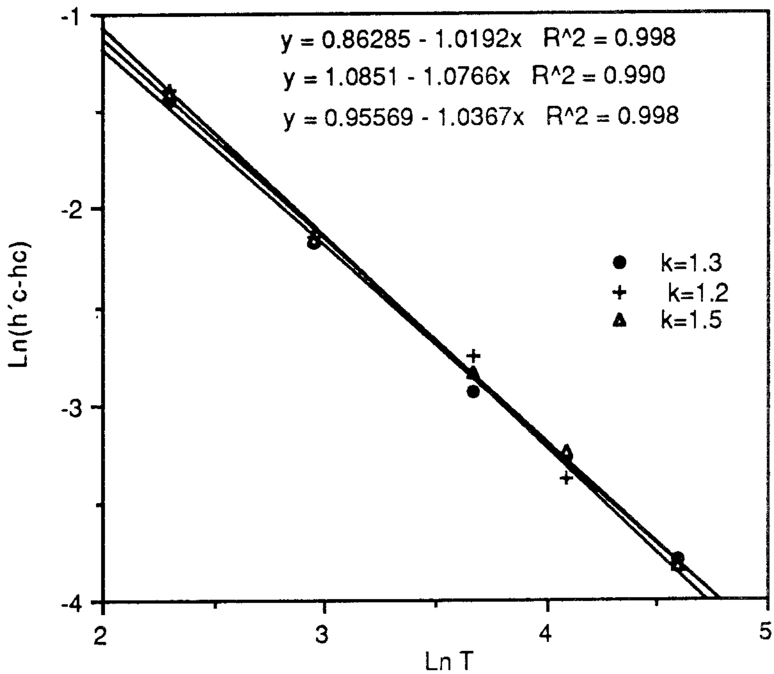


Fig. 9. Variation of $\ln(h'_c - h_c)$ with $\ln T$ for different values of k (1.2, 1.3, 1.5).

h_C, the free energy presents two minima, of which one has lower free energy (corresponding to the preferred phase). However, in the absence of noise, the system may be found in the other minimum (metastable state) until $h = h_C$; when this minimum disappears, the system then "falls" into the stable minimum. However, if the field is changed periodically, then the system may have no time to evolve to the stable point. As the field decreases it may create again the metastable minimum before the system has had the chance to leave the corresponding basin of attraction. Therefore, one expects that if a periodic field is applied, the amplitude of the field that completely destabilizes one phase should increase. This is as observed, through we have no explanation for the simple linear law we obtained.

4.2. Potts Model ($q = 3$)

In this case a sinusoidal external rotating magnetic ac field periodically gives more weight to one of the phases and destabilizes the other two. The applied ac magnetic field has the form

$$h_\alpha(t) = h_0 \sin\left(\omega t + \frac{2\pi\alpha}{q}\right) \tag{17}$$

This is the simplest generalization of the field considered for the Ising case. The phase shift is introduced to force the field to rotate in the phase space, so that the ac magnetic field points periodically along the symmetry directions of the phase space. If the field amplitude h_0 is small, the system evolves toward the basins of attraction of one of the stable fixed points; but for higher values we observe stationary cycles involving the three fixed points. The most interesting case is when the temperature is near the first-order phase transition point (Fig. 1d): at fixed value of T and varying the magnitude of the ac field, we observe a cascade of period-doubling bifurcations. Figures 10a–10d show a typical cycle in each of the first intervals of bifurcation. This happens because the system is forced by the field to leave one minimum of the free energy and pass to another one, after, however, some time delay (of the order of the relaxation time). This time delay is independent of the applied field, so that when the system finds itself near the minimum previously defined by the field, this has already taken an

(a)

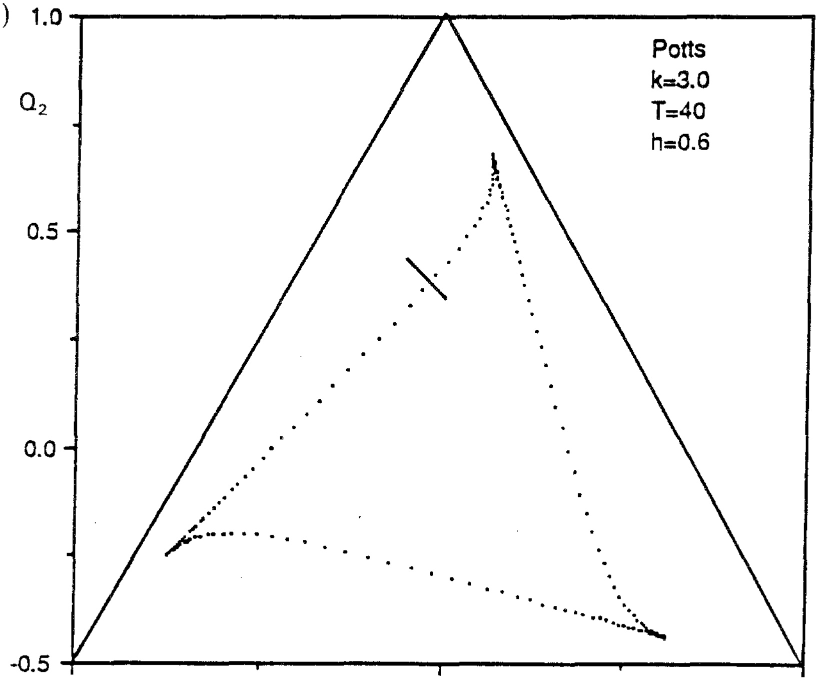


Fig. 10. Representative trajectories in the $q=3$ Potts phase space with parameter values as indicated. These trajectories change from small periods to high periods when the field h_0 changes. T is the period of the applied field; T' is the period of the induced order parameter. (a) $T'=T$ $(=40)$; (b) $T'=2T$; (c) $T'=4T$; (d) $T'=8T$.

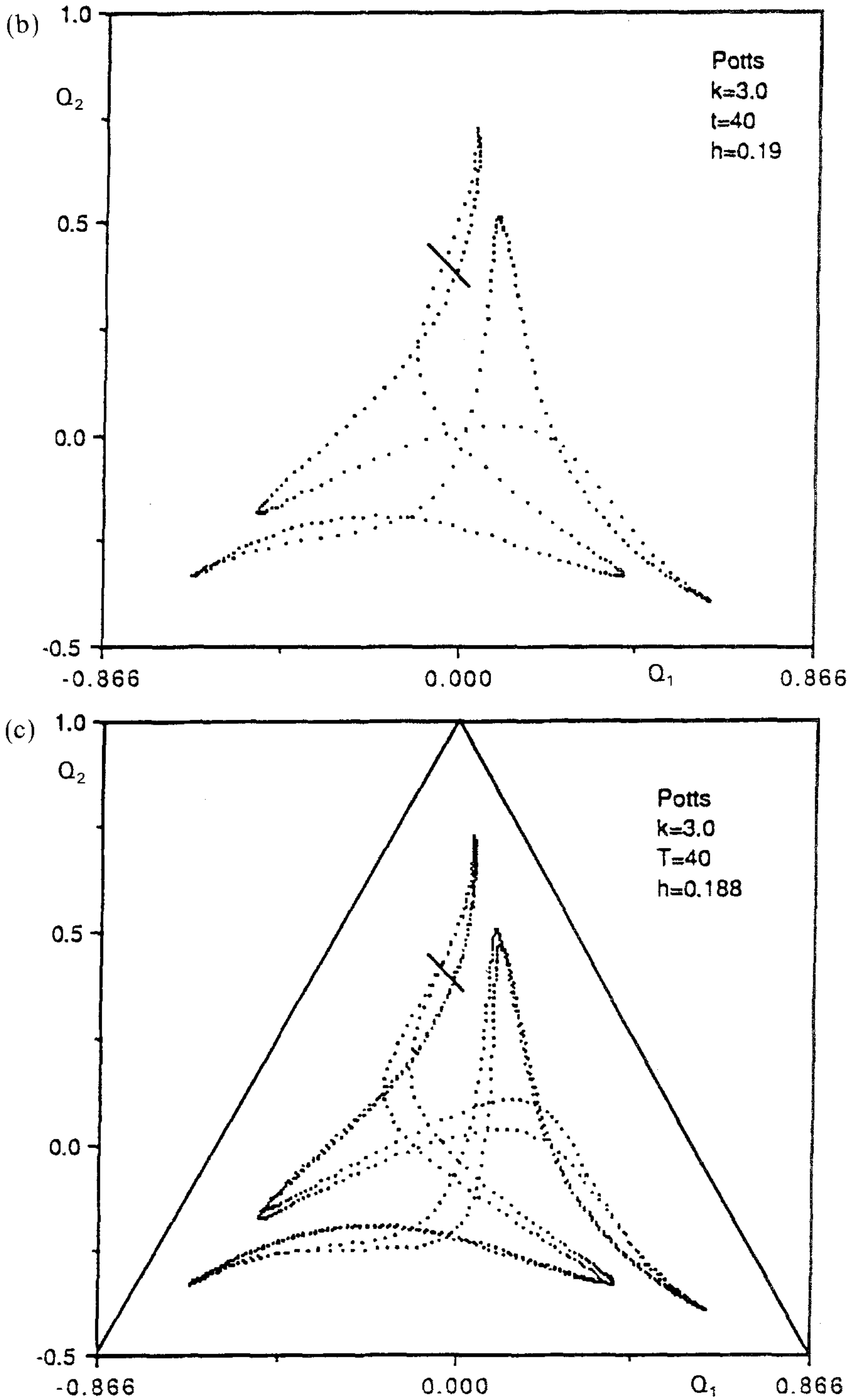


Fig. 10. *(Continued)*

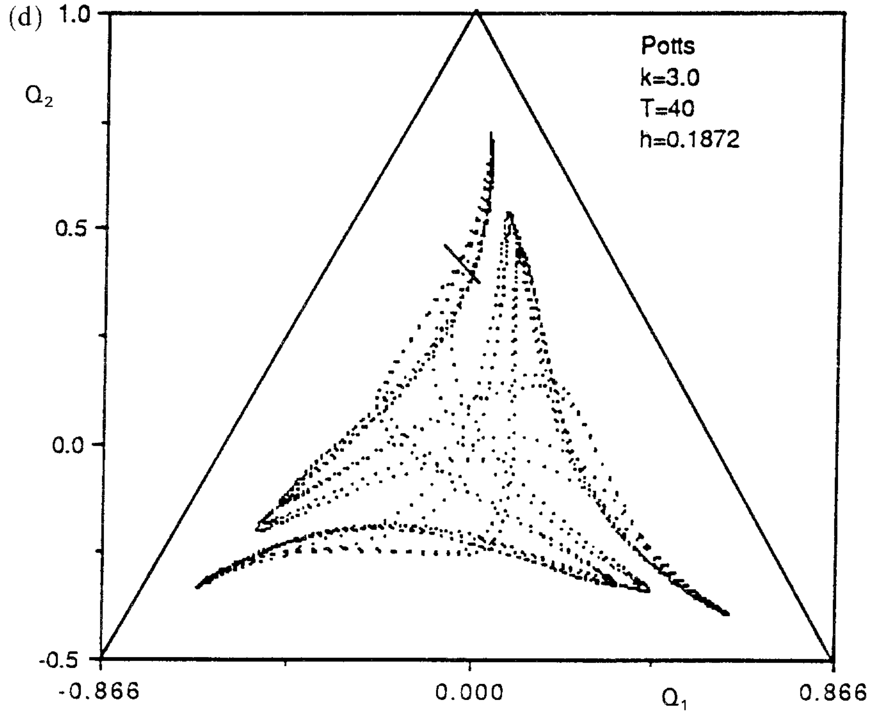


Fig. 10. (*Continued*)

unfavorable value. It appears to us that this type of behavior (found for $q=3$, but not found for the Ising model) is related to the topology of phase space: the attractors are separated by high-energy barriers for certain directions (along which the transverse modes exhibit their fast decay) and lower energy barriers for perpendicular directions (along which the longitudinal mode relaxes). Thus, after an intial transient period, the system settles in a dynamic state characterized by the conflict between the slow relaxational mode, driving the system toward equilibrium, and the external field, dragging the system away from the attractors. A numerical study of this behaviour allows us to estimate Feigenbaum's universal number δ.[2,3] We find $\delta = 4.6 \pm 0.2$. We obtain the values of the field amplitude h_0 for which the period duplicates by analyzing the order parameters $Q_1(t)$ and $Q_2(t)$. We use the Runge-Kutta method of integration of equations and the power spectra method, and we compute the Fourier transform by using the fast Fourier transformation method (FFT) of the solution trajectories with 2048 and 4096 points. Standard routines are used (the routine for FFT can be seen in detail in ref. 7). The rapid convergence rate of the doubling sequence makes it very difficult to observe many bifurcations and is responsible for the numerical error.

5. CONCLUSION

We study single-spin-flip dynamics in the infinite-range q-state Potts model, where we derive the Langevin equation from the microscopic master equation. We first study evolution toward stable and metastable states (for $q \geqslant 3$). We show that the $q-1$ modes separate into fast ones (transverse) and a slow one (longitudinal) associated with order parameter relaxation. We also consider the effects induced by a sinusoidally varying external magnetic field.

For the Ising case, we found a dynamic transition with a characteristic frequency dependence of the spinodal field below the Curie point with a sharp increase at the transition point. Analyzing the amplitudes of first and second harmonics, we extracted the value h_c' where this sharp increase takes place. We found that

$$h_c' - h_c \propto \frac{1}{T} \propto \omega$$

where h_c is the spinodal (static) field.

For $q = 3$ we observe limit cycles and a cascade of period-doubling bifurcations when the field magnitude changes. We numerically estimate Feigenbaum's δ exponent as $\delta = 4.6 \pm 0.2$.

APPENDIX

Here we derive Eq. (6) in a more detailed form. Using the rates (3), we can write the master Eq. (2) as

$$\frac{\partial P(\sigma_i, \{\sigma\}_i; t)}{\partial t} = \sum_i \sum_{\{\sigma_i'\}} (1 - q\delta_{\sigma_i \sigma_i'}) \, W(\sigma_i') \, P(\sigma_i', \{\sigma\}_i; t) \qquad \text{(A.1)}$$

Introducing the variables x_σ and the probabilities $P(\mathbf{x}; t)$ $[\mathbf{x} = (x_1, x_2, ..., x_q)]$ given by (5), we can obtain the following equation:

$$\frac{\partial P(\mathbf{x}; t)}{\partial t} = \sum_i \sum_{\sigma_i} \left[\sum_{\sigma_i'} (1 - q\delta_{\sigma_i, \sigma_i'}) \, W(\sigma_i') \, P_1 \left(\sigma_i', x_1 - \frac{1}{N} \delta_{\sigma_i, 1}, x_q - \frac{1}{N} \delta_{\sigma_i, q}; t \right) \right] \qquad \text{(A.2)}$$

where

$$P_1\left(\sigma_i', x_1-\frac{1}{N}\delta_{\sigma_i,1},..., x_q-\frac{1}{N}\delta_{\sigma_i,q}; t\right)$$
$$=\sum_{\{\sigma\}_i} P(\sigma_i', \{\sigma\}_i; t)) \prod_{\beta=1}^{q} \delta_{x_\alpha-(1/N)\delta_{\sigma_i,\alpha}, (1/N)\sum_{j\neq 1}\delta_{j,\alpha}} \tag{A.3}$$

That is, P_1 is the probability distribution of one spin in a selected state and all the others distributed with fractions $x_1, x_2,..., x_q$.

We now make an expansion in powers of $1/N$ ($N\to\infty$) of Eq. (A.2), obtaining the following Fokker–Planck equation:

$$\frac{\partial P(\mathbf{x}; t)}{\partial t}=-\sum_\alpha \frac{\partial}{\partial x_\alpha}\left[\sum_{\sigma'}(1-q\delta_{\sigma',\alpha})\, W(\sigma')\, P_1(\sigma', \mathbf{x}; t)\right] \tag{A.4}$$

We easily find from the definitions of the probability densities that

$$P(\sigma_i, \mathbf{x}; t)=\sum_{\{\sigma\}_i} P(\sigma_i, \{\sigma\}_i; t)) \prod_{\alpha=1}^{q} \delta_{x_\alpha-(1/N)\delta_{\sigma_i,\alpha}, (1/N)\sum_{j\neq i}\delta_{\sigma_j,\alpha}}$$
$$=F_{\sigma_i}(\mathbf{x}; t) \tag{A.5}$$

obey the normalization condition:

$$\sum_\alpha \int\left(\prod_\gamma dx_\gamma\right) F_\alpha(\mathbf{x}; t)\delta\left(\sum_{\sigma=1}^{q} x_\sigma-1\right)=1 \tag{A.6}$$

Recalling that the probability to find one spin in state σ is x_σ:

$$x_\sigma(t)=\sum_\alpha P(\alpha; t)\delta_{\sigma,\alpha} \tag{A.7}$$

where $P(\alpha; t)$ is the probability for the system to be in the α state, we easily find

$$P_1(\sigma, \mathbf{x}; t)=x_\sigma P(\mathbf{x}; t) \tag{A.8}$$

Therefore, we obtain

$$\frac{\partial P(\mathbf{x}; t)}{\partial t}=-\sum_\alpha \frac{\partial}{\partial x_\alpha}\left[\sum_\sigma (1-q\delta_{\sigma\alpha})\, W(\sigma)\, x_\sigma P(\mathbf{x}; t)\right] \tag{A.9}$$

This equation has a deterministic evolution, with a probability density given by

$$P(\mathbf{x}; t)=\prod_{i=1}^{q} \delta(x_i-x_i(t))\equiv\delta(\mathbf{x}-\mathbf{x}(t)) \tag{A.10}$$

Using (A.8) and (A.9), we arrive at Eq. (6), which is our final result.

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REFERENCES

1. H. C. Bolton and C. A. Leng, *J. Phys. C: Solid State Phys.* **6**:3737 (1973).
2. M. J. Feigenbaum, *J. Stat. Phys.* **19**: 25 (1978).
3. M. J. Feigenbaum, *J. Stat. Phys.* **21**:669 (1979).
4. G. Forgacs, S. T. Chui, and H. C. Frish, *Phys. Rev. B* **22**:415 (1980).
5. R. J. Glauber, *J. Math. Phys.* **4**:294 (1963).
6. R. B. Griffiths, C. Y. Weng, and C. Y. Langer, *Phys. Rev.* **149**:301 (1966).
7. R. J. Higgins, *Am. J. Phys.* **44**:766 (1976).
8. E. J. S. Lage, *J. Phys. A: Math. Gen.* **18**:2289 (1985).
9. R. B. Potts, *Proc. Camb. Phil. Soc.* **48**:106 (1952).
10. H. Suzuki and R. Kubo, *J. Phys. Soc.* (Japan) **24**:51 (1968).
11. N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
12. P. O. Weir and J. M. Kosterlitz, *Phys. Rev. B* **33**:391 (1986).
13. F. Y. Wu, *Rev. Mod. Phys.* **54**:235 (1982).